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[redacted]

CES Production Function

Summary :

The CES function is a favorite tool for modeling production (or utility) functions. Here we demonstrate that some other well known production functions are special cases of the CES type.

If $Y\left(r_K, r_L\right)$ is a production function with two factors, we can define the **elasticity of substitution** σ as:

$$\sigma = \frac{d}{d\log\left(\frac{MP_L}{MP_K}\right)}\log\left(\frac{r_K}{r_L}\right)$$

with the **marginal productivities**

$$MP_L = \frac{d}{dr_L}Y \qquad \text{and} \qquad MP_K = \frac{d}{dr_K}Y$$

In words: σ measures the relative change of the capital intensity in proportion to the relative change of the reciprocal ratio of the marginal productivities. A high value of σ indicates a narrow substitution between both factors. If $\sigma = 0$, there is no substitution.

Now we introduce a simplified version of a production function with a **c**onstant **e**lasticity of **s**ubstitution (= CES):

CES Production Function: $$Y\left(r_K, r_L\right) = \left[a\cdot r_K^{1-\frac{1}{\sigma}} + (1-a)\cdot r_L^{1-\frac{1}{\sigma}}\right]^{\frac{1}{\left(1-\frac{1}{\sigma}\right)}}$$

with $0 \le a \le 1$ (technological parameter)

r_K : capital

r_L : labor

σ : constant elasticity of substitution

The upper formula yields a division by 0 if σ = 1. We use the symbolic processor of *Mathcad* to compute the limit of the function!

$$\lim_{\sigma \to 1} \left[a \cdot r_K^{1-\frac{1}{\sigma}} + (1-a) \cdot r_L^{1-\frac{1}{\sigma}} \right]^{\frac{1}{\left(1-\frac{1}{\sigma}\right)}} \to r_K^{a} \cdot \frac{r_L}{r_L^{a}}$$

The limit function

$$Y(r_K, r_L) = r_K^{a} \cdot r_L^{1-a}$$

is the well known **Cobb-Douglas** function !

For computational reasons we complete the definition of the CES function:

$$Y(r_K, r_L, a, \sigma) \equiv \begin{cases} r_K^{a} \cdot r_L^{1-a} & \text{if } \sigma = 1 \\ \left[a \cdot r_K^{\frac{\sigma-1}{\sigma}} + (1-a) \cdot r_L^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} & \text{otherwise} \end{cases}$$

<u>Note:</u> For $\sigma = 1$ the function becomes a Cobb-Douglas-function,

$\sigma = 0$ it becomes a Leontief-function,

$\sigma \le 1$ there is no possibility of perfect substitution,

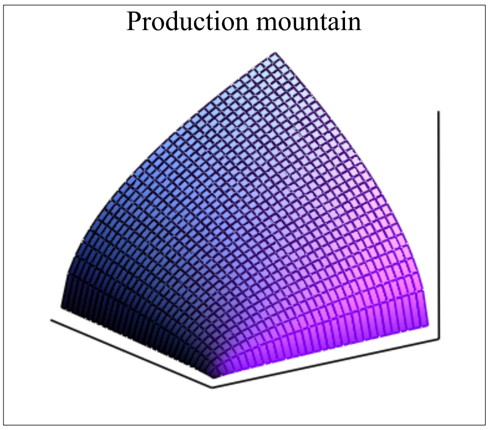
$\sigma > 1$ perfect substitution is possible,

$\sigma = \infty$ it becomes a linear function.

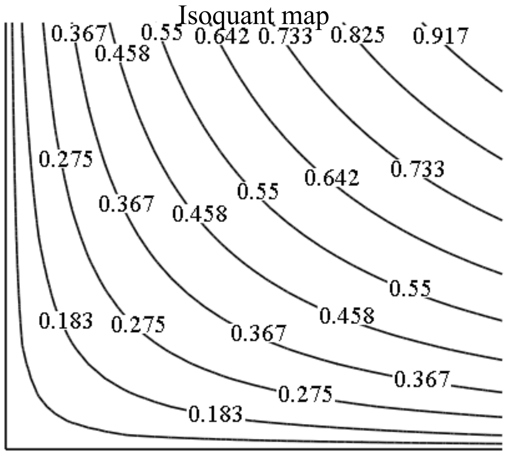
Now it's up to you to do some experiments with the CES function. Change the following parameters and verify the statements above.

Range of plotted values:

$\sigma \equiv 1$ $a \equiv .5$ $r_{Lmax} \equiv 1$ $r_{Kmax} \equiv 1$



Output



Output

<u>Hint:</u> If σ is very small, Mathcad runs into an overflow. For a good approximation of the Leontief function let $\sigma = 0.01$.

Literature :

Arrow, K.J./Chenery, H.B./Minhas, B.S./Solow, R.M.: Capital-Labor Substitution and Economic Efficiency. Review of Economics and Statistics. Vol. 43 (1961), p. 225 - 250.